Multigrid Solver with Super-Resolved Interpolation

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Abstract

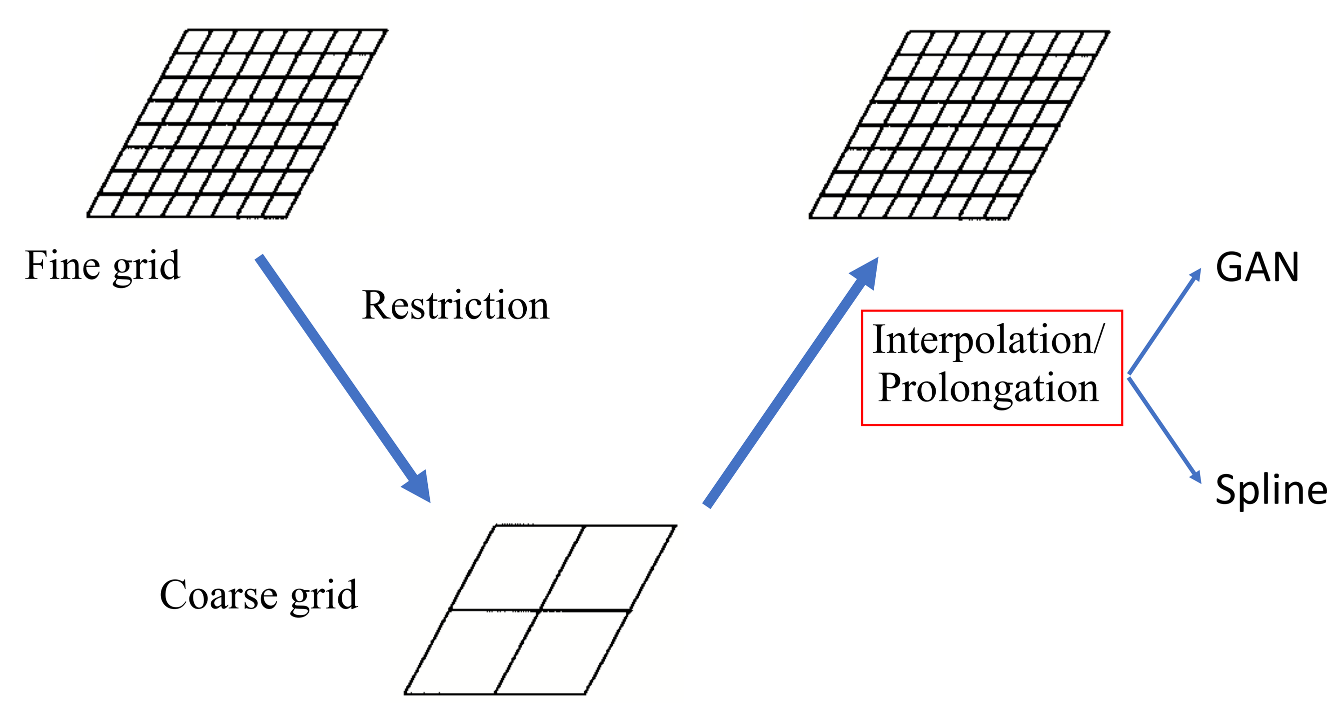
The multigrid algorithm is an efficient numerical method for solving a variety of elliptic partial differential equations (PDEs). The method damps errors at progressively finer grid scales, resulting in faster convergence compared to standard iterative methods. The prolongation, or coarse-to-fine interpolation operator within the algorithm lends itself to a data-driven treatment with ML super resolution. We (i) propose the novel integration of a super resolution generative adversarial network (GAN) model with the multigrid algorithm as the prolongation operator and (ii) show that the GAN-interpolation improves the convergence properties of the multigrid in comparison to cubic spline interpolation on a class of multiscale PDEs typically solved in physics and engineering simulations.

Introduction

- Improving heuristic operators within existing formally derived numerical methods can allow for computational gains, easier implementation, and more rapid deployment in codes.
- Multigrid (MG) methods are attractive due to their ability to efficiently reduce errors at multiple scales.
- The prolongation (interpolation) operator lends itself to a data-driven treatment, as is it similar to super-resolution operators in image analysis.

Multigrid parameters:

$N_{smooth,pre}$: smoothing iterations before solving
N_{smooth}: smoothing iterations between MG steps.
N_{step}: Restriction/interpolation of grid size by 2^(2N_{step}) factor
R_{min}: Side length of coarse grid.



Schematic for a two-level multigrid algorithm. Figure adapted from Chen et al. (2001).

Equations

Physical system and training data

- We focus on the pressure-Poisson formulation of the incompressible Navier-Stokes (NS) equations.
- Using a two-level multigrid method, we solve a Poisson equation for pressure *p* and a source term *f*(*x*,*y*), which is a function of the fluid velocity *u* and viscosity *ν*.

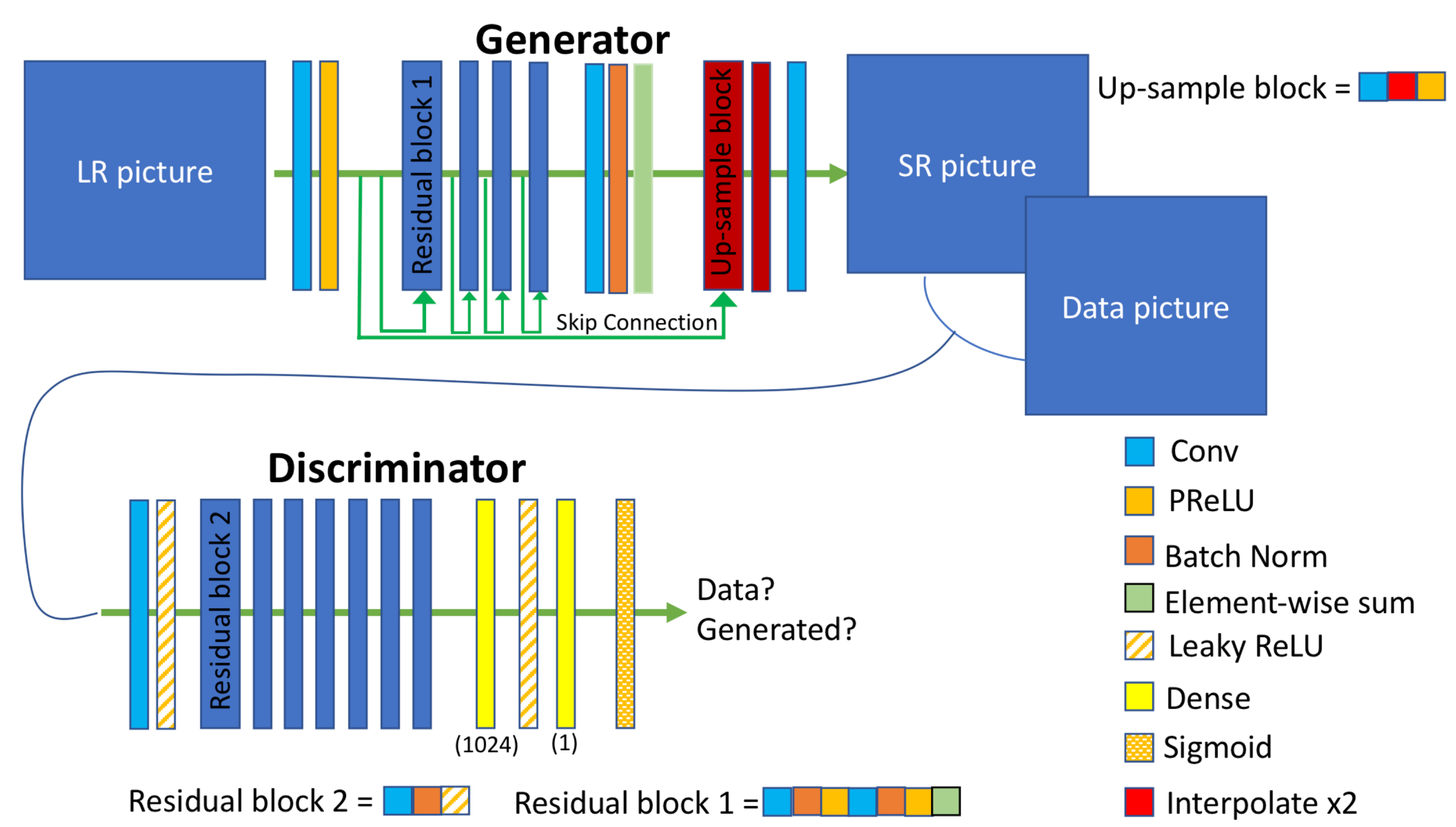
$$\nabla^2 p = \nabla \cdot (\nu \nabla^2 u - (u \cdot \nabla) u) = f(x, y)$$

Proposal

Physical system and training data

- We produce a set of 200 pairs of pressure and source term grids by directly solving the NS equations.
- We select 1000 training grids with random restrictions of the data
- We transform the log of the grid values to the range [-1,1] using the global min and max of all the pressure grids.

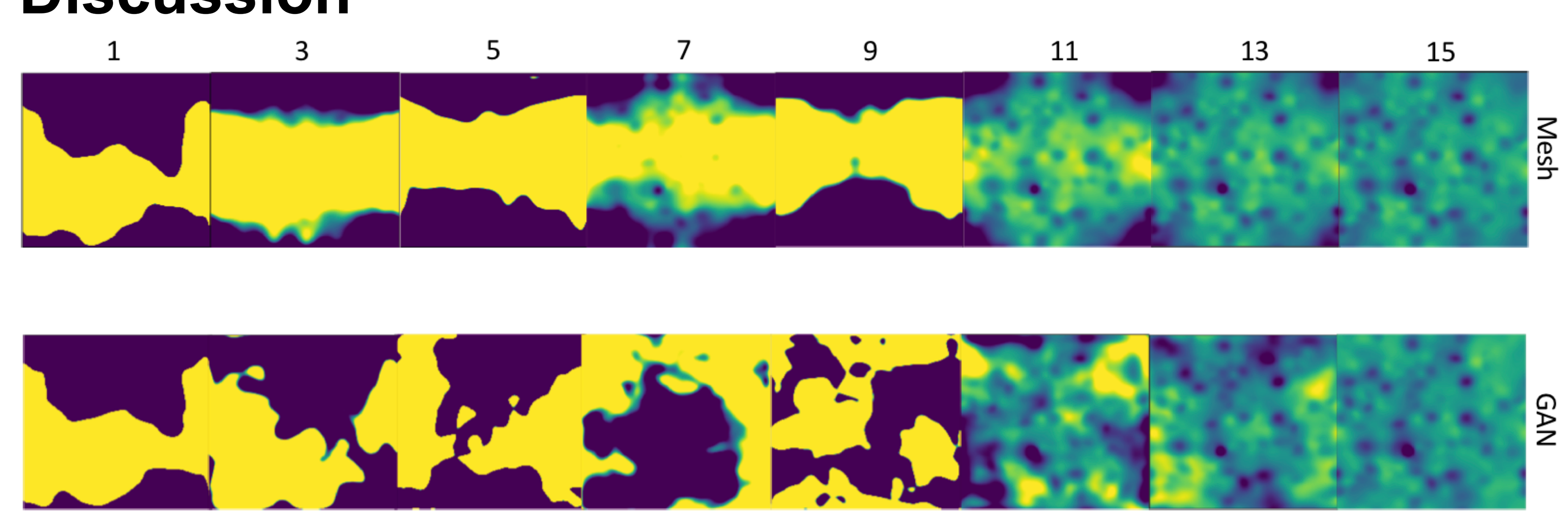
GAN prolongation structure and implementation



Schematic for a super-resolution GAN (see Ledig+2017)

1) Find the max and min data of the grid, and normalize the data to [-1,1], using a symmetric log function.
2) Divide the normalized coarse gird into overlapping windows (n_s^2 kernel, with stride = 2)
3) Apply GAN prolongation operator to produce a set of n_L^2 window kernels.
4) Construct the fine grid by assigning the central $(n_s+2)^2$ window to the fine grid. Any non-overlapping areas are assigned to the appropriate value.

Discussion

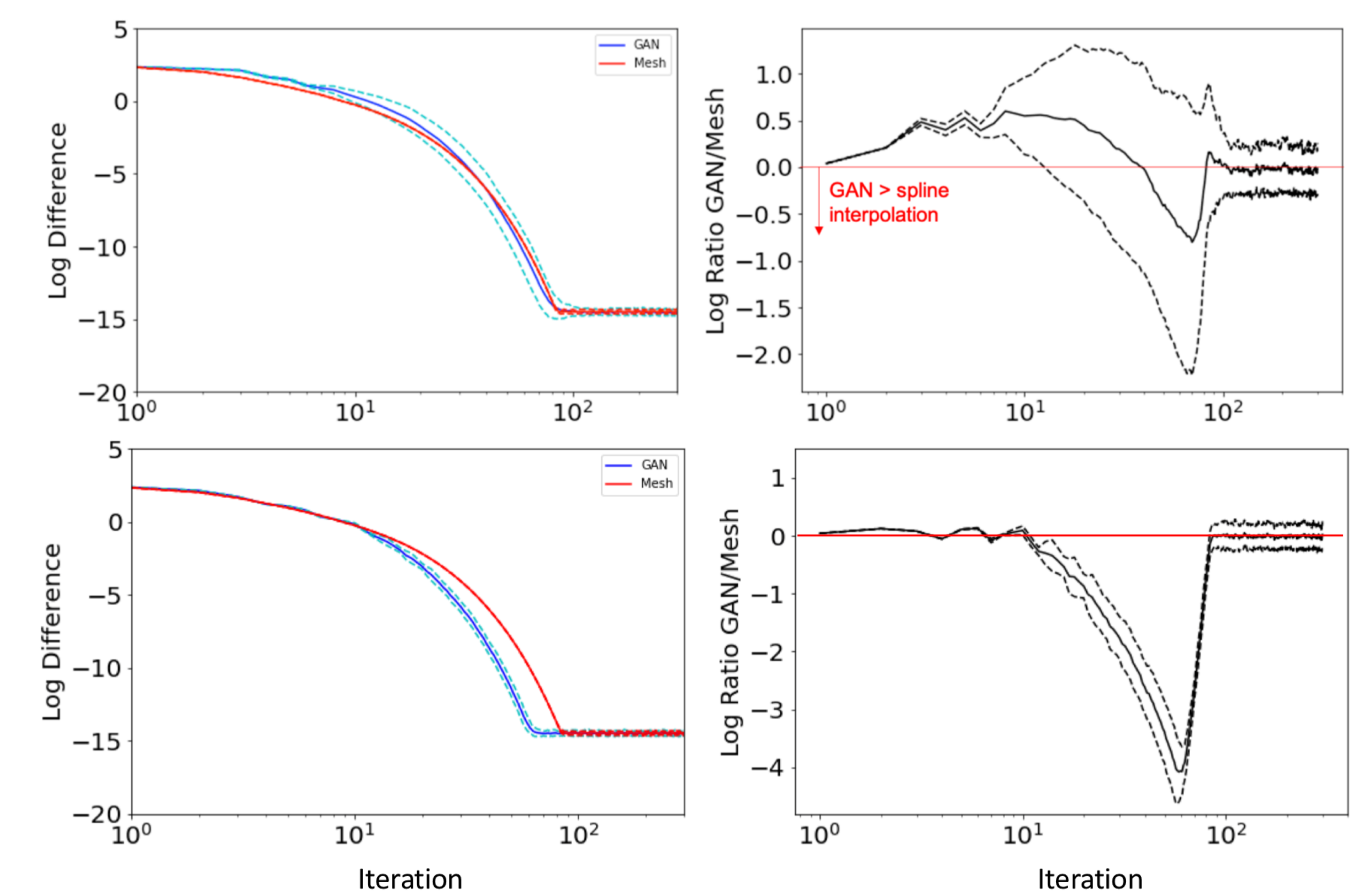


Example of the first 15 iterations of the MG method solving for the pressure.
Top: Interpolation with traditional spline interpolation, Bottom: GAN interpolation

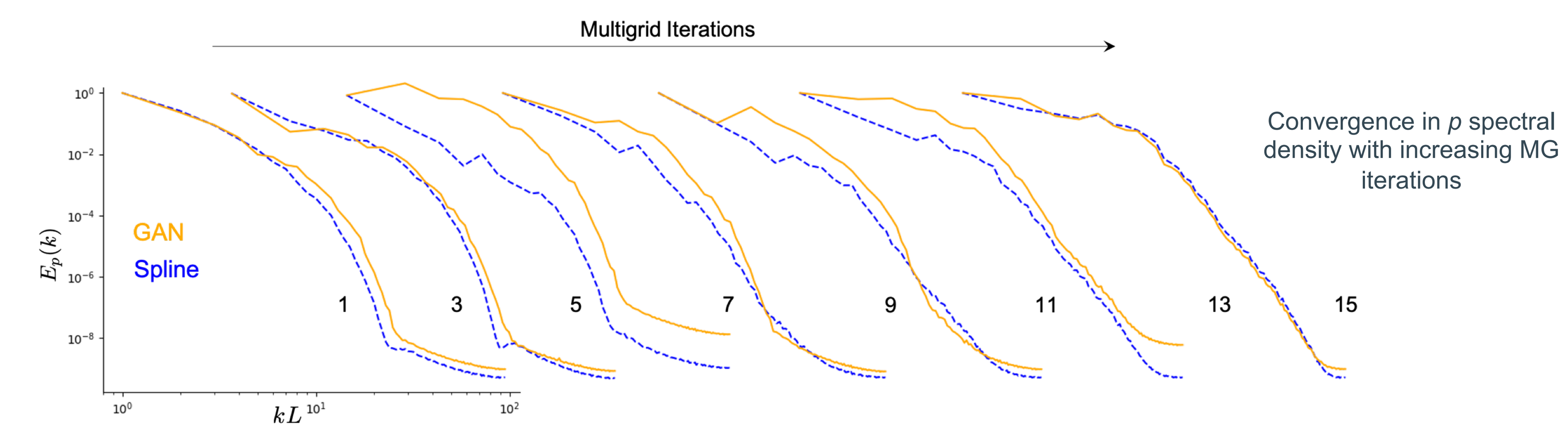
- Both examples of spline vs. GAN-based MG show progression to the same solution.
- Visually, the GAN-based MG solver contains higher frequency information at earlier iterations compared to to the spline-based MG.

Conclusion

Averaged results of MG solving 100 different grids.



Top: GAN operator applied every iteration. Bottom: GAN and spline operators alternated every iteration.
Left: Norm of difference between iterations Right: Ratio between spline and GAN results.



- In this proof-of-concept, the MG method with GAN interpolation on average converged just as fast as with standard interpolation.
- For some grids, the GAN interpolation converged faster than the spline version, but for others, it converged slower.
- With the spline-based interpolation, the MG is unable to capture high to mid-frequency information at early times, while the GAN-based version contains higher frequency information at earlier times.

References

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